Analysis of CDMA systems that are characterized by eigenvalue spectrum

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PACS. 84.40.Ua – Telecommunications: signal transmission and processing; communication satellites.
PACS. 75.10.Nr – Spin-glass and other random models.
PACS. 89.70.+c – Information theory and communication theory.

Abstract. – An approach to analyze the performance of the code division multiple access (CDMA) scheme, which is a core technology used in modern wireless communication systems, is provided. The approach characterizes the objective system by the eigenvalue spectrum of a cross-correlation matrix composed of signature sequences used in CDMA communication, which enable us to handle a wider class of CDMA systems beyond the basic model reported by Tanaka in *Europhys. Lett.,* **54** (2001) 540. The utility of the scheme is shown by analyzing a system in which the generation of signature sequences is designed for enhancing the orthogonality.

Introduction. – Over the last decade, the scope of statistical mechanics has rapidly expanded beyond its original goal of analyzing many-body problems that arise when dealing with material objects. Information theory is a major source of problems, and research activity aimed at solving these problems is becoming popular. Identifying information bits with Ising spins, many problems in information theory, such as error correcting/compression codes [1–10] and cryptosystems [11, 12], can be formulated as virtual many-body systems that are subject to disordered interactions. Analysis of the formulated problems using techniques of statistical mechanics has provided various nontrivial results that have not been obtained by conventional methods of information theory [13, 14].

Code division multiple access (CDMA), which is a core technology used in modern wireless communication, is an example of the successful application of such a statistical mechanical approach. This technology realizes simultaneous communication between multiple users and a single base station by modulating each user's bit signal (symbol) into a sequence of random pattern, termed the signature sequence [15]. CDMA has already been employed in third-generation mobile phone systems and wireless LANs.

Tanaka (2001) showed that the replica method of statistical mechanics enables the accurate assessment of the communication performance of a basic CDMA model in which users' sequences are generated independently of each other in a large system limit [16,17]. This research

was considered to be ground-breaking, and variations of the basic model are currently being actively analyzed based on the scheme given by Tanaka [18–21]. However, Tanaka's scheme relies on the assumption of statistical independence among users' sequences and, therefore, cannot be applied to systems in which sequences are not independent. As certain dependence generally arises among sequences when the system is designed for optimizing communication performance, this limitation is problematic with respect to practical relevance.

The purpose of this Letter is to resolve this problem. More precisely, we present another approach by which to analytically assess the performance of CDMA communication. The central concept of our scheme is to characterize the objective system by the eigenvalue spectrum of the cross-correlation matrix composed of sequences under the assumption that the orthogonal eigenbasis is randomly provided. The range of applicability of such characterization is not clear at the current moment; however, this enables us to analytically handle CDMA systems regardless of whether sequences are statistically independent and offers, at least, a useful approximation scheme for investigating advanced systems. The utility of the approach is shown by analyzing a system that is difficult to assess accurately by the conventional scheme.

Modeling. – As a general scenario of CDMA systems, let us assume a situation in which K users simultaneously transmit symbols $b_k^0 = \pm 1$ ($k = 1, 2, \ldots, K$) to a single base station utilizing the sequences of N-dimensional vectors $\boldsymbol{s}_k = (s_{\mu k})$ ($\mu = 1, 2, \ldots, N$), where μ indexes the chip timing of the symbol modulation and normalization constraint $|\boldsymbol{s}_k| = 1$ is imposed. We denote an $N \times K$ matrix composed of $\boldsymbol{s}_k$ and a set of the received amplitudes of users' signals as $\boldsymbol{S} = (\boldsymbol{s}_k)$ and a diagonal matrix $\boldsymbol{A} = \mathrm{diag}(A_k)$, respectively, both of which are assumed to be known to the base station. In order to cope with cases in which the sequences correlate with each other, we characterize the system by a $K \times K$ cross-correlation matrix $\boldsymbol{R} = \boldsymbol{A}^T\boldsymbol{S}^T\boldsymbol{S}\boldsymbol{A}$, where $\boldsymbol{X}^T$ represents the transpose of $\boldsymbol{X}$, and *assume* that $\boldsymbol{R}$ can be regarded as

$$\boldsymbol{R} = \boldsymbol{O}\boldsymbol{D}\boldsymbol{O}^{-1}, \tag{1}$$

where $\boldsymbol{D}$ is a diagonal matrix $\boldsymbol{D} = \mathrm{diag}(D_k)$, the eigenvalue spectrum of which, $K^{-1}\sum_{k=1}^{K}\delta(\lambda - D_k)$, is provided as $\rho(\lambda)$ and a $K \times K$ orthogonal matrix $\boldsymbol{O}$ is assumed to be drawn randomly from the uniform distribution defined by the Haar measure on the orthogonal group [22]. Such characterization is supposed to be plausible because for a wide class of systems $\boldsymbol{R}$ becomes dense and the sequences are generated in a somewhat random manner, which statistically produces no preferential direction, although mathematical validity of assuming Eq. (1) should be examined in future research. Under the additional simplifying assumptions that the channel noise is the additive white Gaussian of variance σ_0^2 and both the chip timing and symbol timing are synchronous across all users, the received signals at base station, $\boldsymbol{r} = (r_\mu)$, can be represented as

$$\boldsymbol{r} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{b}^0 + \sigma_0\boldsymbol{\eta}, \tag{2}$$

where $\boldsymbol{b}^0 = (b_k^0)$ and $\boldsymbol{\eta}$ is an N-dimensional random vector, each component of which is generated independently from the normal distribution $\mathcal{N}(0, 1)$.

After receiving $\boldsymbol{r}$, the remaining task of the base station is to infer the users' original symbols $\boldsymbol{b}^0$, which is called user detection or demodulation. The optimal demodulation scheme to minimize the bit-wise probability of incorrect estimation, which is referred to as the bit error rate P_b, is provided by using the posterior distribution

$$P(\boldsymbol{b}|\boldsymbol{r}) = Z^{-1}\exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{r} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{b})^T(\boldsymbol{r} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{b})\right], \tag{3}$$

as $\hat{b}_k = \mathrm{sign}\left(\sum_{\boldsymbol{b}} b_k P(\boldsymbol{b}|\boldsymbol{r})\right)$ when the assumed noise variance σ^2 is in accordance with the correct value σ_0^2. Here, $Z = \sum_{\boldsymbol{b}} \exp\left[-(\boldsymbol{r} - \boldsymbol{SAb})^T(\boldsymbol{r} - \boldsymbol{SAb})/(2\sigma^2)\right]$ serves as the partition function, $\hat{b}_k$ denotes the estimate of the kth user's symbol b_k^0 and $\mathrm{sign}(u) = u/|u|$ for $u \neq 0$.

Analysis. – Since Eq. (3) is conditioned by the predetermined random variable $\boldsymbol{r} = \boldsymbol{SAb}^0 + \sigma_0\boldsymbol{\eta}$, we resort to the replica method to assess the typical performance of the CDMA system assuming a large system limit $K, N \to \infty$ while maintaining the load $\beta = K/N$. Namely, we evaluate the $n(= 1, 2, \ldots)$th moment of the partition function Z with respect to the relevant predetermined randomness and analytically continue the obtained expression to real $n \in \mathbf{R}$. For this purpose, it is convenient to first take the average with respect to $\boldsymbol{\eta}$, which provides

$$\int D\boldsymbol{\eta} \exp\left[-\frac{1}{2\sigma^2}\sum_{a=1}^{n}(\boldsymbol{r} - \boldsymbol{SAb}^a)^T(\boldsymbol{r} - \boldsymbol{SAb}^a)\right] = \exp\left[\frac{1}{2}\mathrm{Tr}\boldsymbol{RL}(n)\right], \tag{4}$$

where $\boldsymbol{b}^a$ is the ath replica of symbol vector $\boldsymbol{b}$, $D\boldsymbol{\eta} = (2\pi)^{-N/2}\exp\left[-|\boldsymbol{\eta}|^2/2\right]$ and

$$\boldsymbol{L}(n) = -\frac{1}{\sigma^2}\sum_{a=1}^{n}(\boldsymbol{b}^0 - \boldsymbol{b}^a)(\boldsymbol{b}^0 - \boldsymbol{b}^a)^T + \frac{\sigma_0^2}{\sigma^2(\sigma^2 + n\sigma_0^2)}\left(\sum_{a=1}^{n}(\boldsymbol{b}^0 - \boldsymbol{b}^a)\right)\left(\sum_{b=1}^{n}(\boldsymbol{b}^0 - \boldsymbol{b}^b)\right)^T. \tag{5}$$

Equation (1) indicates that the average of Eq. (4) with respect to the cross-correlation matrix $\boldsymbol{R}$ is reduced to a form of

$$\int \mathcal{D}\boldsymbol{R} \exp\left[\frac{1}{2}\mathrm{Tr}\boldsymbol{RL}(n)\right] \simeq \exp\left[K\mathrm{Tr}G\left(\frac{\boldsymbol{L}(n)}{K}\right)\right], \tag{6}$$

for large K, where a function $G(x)$ is provided by the eigenvalue spectrum $\rho(\lambda)$ as $G(x) = (1/2)\int_0^x dt\left(\Lambda(t) - t^{-1}\right)$, utilizing a function $\Lambda(x)$ that is implicitly determined by the condition [22, 23]

$$x = \int d\lambda \rho(\lambda)(\Lambda(x) - \lambda)^{-1}. \tag{7}$$

For a given G-function, $G(x)$, Eq. (7) (Cauchy transform) can also be utilized to assess x as a function of Λ. The function $x(\Lambda)$ is directly linked to the spectrum by Stieltjes inversion formula,

$$\rho(\lambda) = \lim_{\epsilon \to +0}\frac{1}{\pi}\mathrm{Im}\, x(\lambda - \sqrt{-1}\epsilon). \tag{8}$$

Intrinsic permutation symmetry among replicas naturally leads to the replica symmetric (RS) ansatz, implying that configurations characterized by $\boldsymbol{b}^0 \cdot \boldsymbol{b}^a = Km$ $(a = 1, 2, \ldots, n)$ and $\boldsymbol{b}^a \cdot \boldsymbol{b}^b = Kq$ $(a > b)$ provide the most dominant contribution to the moment evaluation. Under this ansatz, the $K \times K$ matrix $\boldsymbol{L}(n)$ has three types of eigenvalues: $\lambda_1 = -K(\sigma^2 + n\sigma_o^2)^{-1}(1 - q + n(1 - 2m + q))$, $\lambda_2 = -K\sigma^{-2}(1 - q)$ and $\lambda_3 = 0$, the numbers of degeneracy of which are 1, $n - 1$ and $K - n$, respectively. This indicates that Eq. (6) is evaluated as

$$\exp\left[K\left(G\left(-\frac{1 - q + n(1 - 2m + q)}{\sigma^2 + n\sigma_o^2}\right) + (n - 1)G\left(-\frac{1 - q}{\sigma^2}\right)\right)\right]. \tag{9}$$

Continuing both of this and the RS entropic contribution of $\mathrm{Tr}_{\boldsymbol{b}^1,\boldsymbol{b}^2,\ldots,\boldsymbol{b}^n}\prod_{a=1}^n \delta(\boldsymbol{b}^0\cdot\boldsymbol{b}^a - Km)\prod_{a>b}\delta(\boldsymbol{b}^a\cdot\boldsymbol{b}^b - Kq)$ analytically from $n=1,2,\ldots$ to real values, $n\in\mathbf{R}$, provides an expression of the average free energy

$$\begin{aligned}\frac{1}{K}\overline{\ln Z} &= \lim_{n\to 0}\frac{1}{nK}\ln\overline{Z^n} \\ &= \mathop{\mathrm{Extr}}_{m,q,\hat{m},\hat{q}}\left\{G\left(-\frac{1-q}{\sigma^2}\right)+\left(-\frac{1-2m+q}{\sigma^2}+\frac{\sigma_0^2(1-q)}{\sigma^4}\right)G'\left(-\frac{1-q}{\sigma^2}\right)\right. \\ &\qquad \left. -\hat{m}m-\frac{\hat{q}(1-q)}{2}+\int Dz\ln\left(2\cosh\sqrt{\hat{q}}z+\hat{m}\right)\right\}, \end{aligned} \tag{10}$$

where $Dz=(2\pi)^{-1/2}dz\exp\left[-z^2/2\right]$, $\mathrm{Extr}_u\{\cdots\}$ indicates the extremization of $\{\cdots\}$ with respect to u, and $\overline{(\cdots)}$ denotes the average of $(\cdots)$ with respect to $\boldsymbol{\eta}$ and $\boldsymbol{R}$. Equation (10) is the main result of the present letter.

Three things are noteworthy here. First, the bit error rate P_b can be assessed as $P_b=\int_{\hat{m}/\sqrt{\hat{q}}}^{\infty}Dz$ by utilizing the values of the conjugate variables $\hat{q}$ and $\hat{m}$ that extremize Eq. (10). This implies that Eqs. (7) and (10) offer a complete scheme for assessing the typical communication performance of CDMA systems characterized by the eigenvalue spectrum $\rho(\lambda)$. Second, the basic model analyzed in [16] can be characterized by $\rho(\lambda)=\rho_{\mathrm{basic}}(\lambda)=[1-\beta^{-1}]^+\delta(\lambda)+(2\pi\beta\lambda)^{-1}\sqrt{\left[\lambda-(1-\sqrt{\beta})^2\right]^+\left[(1+\sqrt{\beta})^2-\lambda\right]^+}$, where $[u]^+=u$ for $u>0$ and 0, otherwise. This yields $G(x)=G_{\mathrm{basic}}(x)=-(2\beta)^{-1}\ln(1-\beta x)$, which, when inserted into Eq. (10), reproduces an expression equivalent to the free energy of [16] (Eq. (10) in [16]). Namely, the conventional analysis is one component of our formalism. Finally, although thus far we have assumed the RS ansatz, the local stability of the RS solution can be broken when the assumed noise variance σ^2 is set sufficiently smaller than the correct value σ_0^2. The stability condition with respect to the local perturbation of breaking the replica symmetry can be examined via the de Almeida-Thouless analysis [24], which, in the current case, yields the following expression:

$$1-\frac{2}{\sigma^4}G''\left(-\frac{1-q}{\sigma^2}\right)\int Dz\left(1-\tanh^2(\sqrt{\hat{q}}z+\hat{m})\right)^2>0. \tag{11}$$

A similar result was reported in [23].

Computationally feasible demodulation algorithm. – Although the above mentioned procedure provides the typical performance of the estimator $\hat{b}_k=\mathrm{sign}\left(\sum_{\boldsymbol{b}}b_kP(\boldsymbol{b}|\boldsymbol{r})\right)$, performing the procedure exactly for a given received signal $\boldsymbol{r}$ is computationally difficult in the general case. Advanced mean field methods of the Thouless-Anderson-Palmer (TAP) type have recently been receiving attention as promising approaches by which to resolve this difficulty [25,26]. For the basic models, one can construct a computationally feasible demodulation algorithm along this line by using the property of statistical independence of sequences [27]. Unfortunately, this strategy is not available in the present case. Nevertheless, one can construct such a heuristic algorithm that coincides with the result for the basic model when $\rho(\lambda)$ is set to $\rho_{\mathrm{basic}}(\lambda)$. Denoting the approximate symbol average $\sum_{\boldsymbol{b}}b_kP(\boldsymbol{b}|\boldsymbol{r})$ at the tth update as $m_k^t=\tanh(h_k^t)$, we obtain

$$\begin{aligned}\boldsymbol{a}^{t+1} &= 2G'\left(-\tfrac{1-q^t}{\sigma^2}\right)\left[\tfrac{1}{\sigma^2}\left(\boldsymbol{r}-\boldsymbol{SAm}^t\right)+\left(\frac{1}{2G'\left(-\frac{1-q^t}{\sigma^2}\right)}-1\right)\boldsymbol{a}^t\right], \\ \boldsymbol{h}^t &= \boldsymbol{A}^T\boldsymbol{S}^T\boldsymbol{a}^t+\tfrac{2}{\sigma^2}G'\left(-\tfrac{1-q^{t-1}}{\sigma^2}\right)\boldsymbol{m}^{t-1},\end{aligned} \tag{12}$$

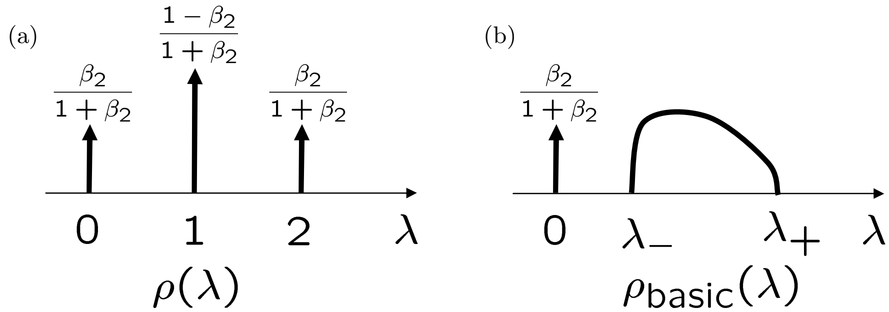

Fig. 1 – Schematic diagrams of eigenvalue spectra for (a) the current example system of $\beta_1 = 1$, $\beta_2 = \beta - 1 \geq 0$ and $\beta_k = 0\ \forall k \geq 3$ and (b) the basic model of [16], in which the component of the sequences, $s_{\mu k}$, is provided as $+N^{-1/2}$ or $-N^{-1/2}$ with an equal probability independently of a pair of indices μ, k. Amplitudes A_k are set to unity $\forall k$ in both systems. Even in the basic model, a randomly drawn pair of sequences, $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, is regarded as *almost* orthogonal for large N since the overlap $\boldsymbol{s}_i \cdot \boldsymbol{s}_k$ typically vanishes as $O(N^{-1/2})$ as $N \to \infty$. However, the constraint of perfect orthogonality within an identical group that is imposed in the example system makes the feature of the eigenvalue spectrum $\rho(\lambda)$, which is composed of three delta-peaks at $\lambda = 0$, 1 and 2, considerably different from that of $\rho_{\mathrm{basic}}(\lambda)$, which is made up of a single delta-peak at $\lambda = 0$ and a continuous part between $\lambda_+ = (1+\sqrt{\beta})^2$ and $\lambda_- = (1-\sqrt{\beta})^2$.

where $\boldsymbol{m}^t = (m_k^t)$, $\boldsymbol{h}^t = (h_k^t)$, $\boldsymbol{a}^t = (a_\mu^t)$ and $q^t = K^{-1}\sum_{k=1}^K (m_k^t)^2$. The estimate of $\hat{b}_k$ at the tth update is provided as $\hat{b}_k = \mathrm{sign}(h_k^t)$. Numerical experiments indicate that Eq. (12) empirically converges by $O(1)$ updates in most cases of large systems when the ratio σ_0^2/σ^2 is sufficiently low. This implies that one can approximately perform the demodulation by serial computers in a time scale of $O(NK)$. Note that Eq. (12) is simply an example of the dynamics, the fixed point of which agrees with the solution of the TAP equation in general, whereas several favorable properties hold in the specific case of $\rho(\lambda) = \rho_{\mathrm{basic}}(\lambda)$ [28]. The quest for better algorithms of this sort is currently under way, as will be reported elsewhere.

Example. – In order to address the significance of the developed approach, we consider a CDMA system in which the generation of sequences is devised for the purpose of enhancing the orthogonality. In this system, each $\boldsymbol{s}_k$ is constructed so as to be orthogonal to all of the existing sequences $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{k-1}$ until k reaches N. Sequence generation of this sort becomes impossible when k exceeds N. Therefore, $\boldsymbol{s}_{N+1}$ is randomly generated and $\boldsymbol{s}_k$ for $N+2 \leq k \leq 2N$ is constructed so as to be orthogonal to $\boldsymbol{s}_{N+1}, \boldsymbol{s}_{N+2}, \ldots, \boldsymbol{s}_{k-1}$ without regard to the sequences of the previous group, $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_N$. For $k \geq 2N+1$, this procedure is repeated. Sequence generation of this sort was once considered with respect to the improvement of the learning performance of linear perceptrons [29]. For $N = 2^L$ $(L = 1, 2, \ldots)$, such generation is easily implemented by drawing a random gauge vector $\boldsymbol{\tau}_g = (\tau_{\mu g}) = \{+1, -1\}^N$ for each group $g (= 1, 2, \ldots)$ of size N and assigning the sequences of group $g = \lfloor k/N \rfloor + 1$ as $\boldsymbol{s}_k = (\tau_{\mu \lfloor k/N \rfloor} e_{\mu \Delta(k,N)})$ by utilizing the normalized Walsh-Hadamard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N$ [30]. Here, $\lfloor u \rfloor$ indicates the maximum integer that does not exceed u, and $\Delta(k, N) = k - N\lfloor k/N \rfloor$. The orthogonality perfectly cancels the crosstalk noise from other users of the identical group, which is advantageous for the improvement of communication performance. However, the orthogonality constraint yields strong dependence among sequences of an identical group, which prevents the scheme provided in [16] from accurately assessing the performance.

On the other hand, our approach potentiates performance analysis by evaluating the eigenvalue spectrum $\rho(\lambda)$ of $\boldsymbol{R}$ under the assumption of Eq. (1), which is expected to hold since

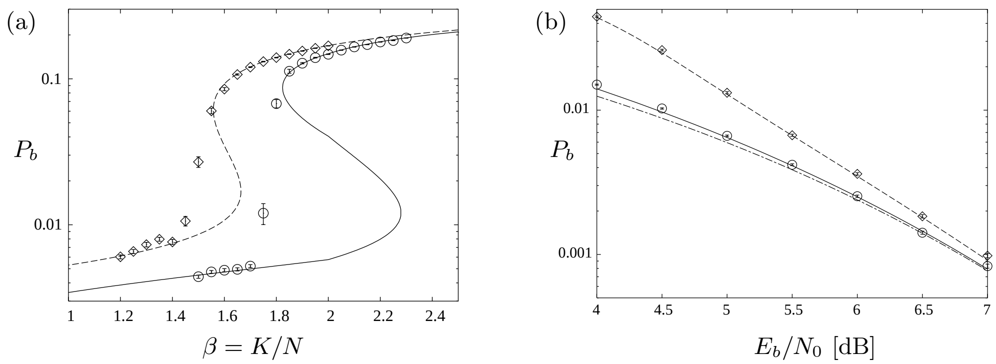

Fig. 2 – (a) Bit error rate P_b vs. the load $\beta = K/N$ under the condition of $\sigma_0 = \sigma = 0.37$. The solid and broken lines indicate the theoretical prediction for the example system and the basic model, respectively. Because of the reduction of the crosstalk noise, a lower bit error rate can be achieved in the example system than that of the basic model for any load β. In addition, the example system provides a larger value of the spinodal point than that of the basic model, which implies that the example system can deal with more users by computationally feasible algorithms than the basic model. The circles and diamonds represent the results obtained from 200 experiments utilizing variations of Eq. (12) for the example system and basic model of $N = 2048$, respectively, in which initial states were set to the matched filter output $h_k^0 = \boldsymbol{s}_k \cdot \boldsymbol{r}$ following the conventional scheme [27,28]. These exhibit excellent agreement with theoretical curves, except around the spinodal points, where large fluctuations make it difficult for the algorithm to converge. (b) Bit error rate P_b vs. $E_b/N_0 = 10\log_{10}[1/(2\sigma^2)]$, which is a conventional measure of the signal-to-noise ratio, for $\beta = 1.1$. Starting from the bottom of the figure, the three curves indicate the achievable limit (performance when the channel is accessed by only a single user), the theoretical predictions of the example system and the basic model, respectively. The circles and diamonds represent the numerical results obtained from 500 experiments on the basis of Eq. (12) for the example system, which exhibits excellent consistency with the theoretical prediction.

the sequence generation statistically creates no directional preference. For this purpose, we fist evaluate another spectrum $\tilde{\rho}(\lambda)$ of $\tilde{\boldsymbol{R}} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{S}^T$, which is dual to $\boldsymbol{R}$ and easier to handle. In CDMA systems, a set of users that communicate with a given base station generally varies in time. Let us denote the set of user indices that remain in group g as $\mathcal{K}(g)$ at the objective moment. We also introduce another notation $\beta_g = |\mathcal{K}(g)|/N$, which implies that $0 \le \beta_g \le 1\ \forall g$ and $\beta = \sum_g \beta_g$. Using these, the dual matrix can be expressed as $\tilde{\boldsymbol{R}} = \sum_g \tilde{\boldsymbol{R}}_g$, where $\tilde{\boldsymbol{R}}_g = \sum_{k\in\mathcal{K}(g)} A_k^2 \boldsymbol{s}_k \boldsymbol{s}_k^T$, the eigenvalue spectrum of which is provided as $\tilde{\rho}_g(\lambda) = (1-\beta_g)\delta(\lambda) + N^{-1}\sum_{k\in\mathcal{K}(g)}\delta(\lambda - A_k^2)$. We express the G-function of $\tilde{\boldsymbol{R}}_g$, which is given by $\tilde{\rho}_g(\lambda)$ via Eq. (7), as $\tilde{G}_g(x)$. Statistical independence across the groups guarantees that the G-function of $\tilde{\boldsymbol{R}}$, $\tilde{G}(x)$, is assessed as $\tilde{G}(x) = \sum_g \tilde{G}_g(x)$. This provides the spectrum $\tilde{\rho}(\lambda)$ through Eq. (8), which yields the spectrum of $\boldsymbol{R}$ as $\rho(\lambda) = [1-\beta^{-1}]^+\delta(\lambda) + \beta^{-1}\tilde{\rho}(\lambda)$.

As a simple but nontrivial example, we examined the case of $\beta_g = 1$, $\beta - \lfloor\beta\rfloor$ and 0 for $g (\ge 0)$ of being less than, equal to and greater than $\lfloor\beta\rfloor + 1$, respectively, and $A_k = 1\ \forall k$, which means $\rho(\lambda) = \left[1-\beta^{-1}\right]^+\delta(\lambda) + \beta^{-1}\left((1-\beta+\lfloor\beta\rfloor)\delta(\lambda-\lfloor\beta\rfloor) + (\beta-\lfloor\beta\rfloor)\delta(\lambda-\lfloor\beta\rfloor-1)\right)$ (Fig. 1). Figure 2 shows that the current system of the random orthogonal sequence generation outperforms the basic model with respect to both the bit error rate and the position of the spinodal point, which practically determines the limitation of computationally feasible demodulation. Numerical experiments based on variations of Eq. (12), the results of which are plotted as symbols, exhibit excellent agreement with the theoretical predictions, which are denoted by the curves, indicating the significance of our approach.

Summary. – In summary, we have presented a novel approach by which to analyze the performance of CDMA communication systems. In the provided scheme, the properties of CDMA systems are characterized by the eigenvalue spectrum of the cross-correlation matrix composed of signature sequences. This enables us to accurately analyze a wide class of CDMA systems, even when signature sequences are dependent on each other. The significance of our approach has been demonstrated by application to a system that cannot be handled by the conventional scheme, the results of which showed excellent agreement with numerical experiments.

Application to various cases and generalization to asynchronous communication will be examined in future studies.

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This study was supported in part by Grant-in-Aids Nos. 17340116 and 18079006 from JSPS/MEXT, Japan (YK).

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